Appendix for Hur and Kondo (2016)

A. Additional Tables and Figures

Table A.4: Foreign Reserves as percent of GDP

	1992–1996	1997–2001	2002–2006	2007–2011	2012–2014
Argentina	5.6	7.9	12.8	13.9	6.6
Brazil	6.6	5.7	7.6	13.4	16.0
Chile	19.9	19.8	17.0	13.4	15.4
China	8.0	14.9	30.5	45.5	40.1
Colombia	9.1	8.9	10.6	9.9	10.9
Czech Republic	16.9	20.2	25.3	19.1	25.5
Egypt	26.3	17.8	19.1	16.4	4.4
Hungary	18.4	20.9	16.5	28.9	33.6
India	4.4	7.4	16.7	19.1	14.8
Indonesia	7.3	16.9	13.4	12.0	12.1
Korea	5.7	14.6	24.5	27.0	28.0
Malaysia	31.5	31.4	45.0	45.4	41.0
Mexico	4.2	6.0	8.1	10.5	14.1
Morocco	12.4	14.6	27.9	25.3	17.3
Pakistan	2.2	2.4	10.5	7.2	2.6
Peru	13.5	16.8	17.0	25.7	29.3
Philippines	8.5	14.1	15.8	24.0	27.5
Poland	7.4	14.7	14.1	16.1	19.6
Romania	5.1	7.6	17.8	22.9	22.6
Russia	3.0	6.1	20.6	29.6	21.3
South Africa	1.1	4.2	6.3	11.0	12.5
Thailand	20.2	25.0	29.8	45.3	43.2
Turkey	4.4	8.4	10.8	10.7	12.8
median	7.4	14.6	16.7	19.1	17.3

Table A.5: Foreign Reserves as percent of External Debt Liabilities

	1992–1996	1997–2001	2002–2006	2007–2011	2012–2014
Argentina	16.2	15.7	17.3	41.6	25.4
Brazil	25.0	17.8	28.2	78.2	74.5
Chile	57.0	52.4	40.2	38.3	41.3
China	50.8	108.0	245.5	438.0	290.9
Colombia	37.3	24.9	33.1	44.6	42.8
Czech Republic	57.2	63.0	82.4	51.3	54.1
Egypt	44.4	51.3	56.9	79.1	
Hungary	32.2	39.6	25.7	27.6	37.6
India	15.4	33.9	94.1	104.0	70.3
Indonesia	11.8	16.5	26.4	44.0	
Korea	27.9	49.0	103.1	74.6	83.6
Malaysia	75.9	57.2	98.4	101.4	72.5
Mexico	12.3	21.1	37.2	42.2	40.0
Morocco	17.4	30.1	93.8	94.6	
Pakistan	5.3	5.5	27.6	22.1	9.9
Peru	20.5	32.5	42.2	90.0	100.1
Philippines	16.1	20.0	24.8	64.1	
Poland	19.3	45.6	36.2	34.4	36.0
Romania	23.5	27.6	53.9	45.0	40.7
Russia	7.3	11.8	60.1	99.7	
South Africa	4.8	16.4	32.5	48.9	40.2
Thailand	39.8	37.8	93.0	178.9	
Turkey	13.7	19.7	25.0	26.2	25.5
median	20.5	30.1	40.2	51.3	41.3

Table A.6: Panel Logit Estimation for Other Crises

	S.D.	Crisis in 1–2 years δp	Crisis in 1–2 years $\frac{\partial p}{\partial x}$	Crisis 1–3 years δp	Crisis 1–3 years $\frac{\partial p}{\partial x}$
Panel A: Default Crises (baseline sample: country FE and years 1990-2011)					
Reserves	20.45	-0.07	-0.02	-0.11	-0.03
over External Debt		(0.16)	(0.04)	(0.29)	(0.07)
Net Foreign Assets	8.91	-0.07	-0.02	-0.10	-0.03
over GDP		(0.15)	(0.04)	(0.27)	(0.07)
Probability in percent ($\overline{p}$)		0.07		0.11	
N=6 ; $N\times T$=98					
Panel B: Banking Crises (baseline sample: country FE and years 1990–2011)					
Reserves	51.28	-5.73***	-0.22***	-8.34***	-0.36***
over External Debt		(1.45)	(0.07)	(1.99)	(0.08)
Net Foreign Assets	12.49	-2.53	-0.25	-2.88	-0.27
over GDP		(1.81)	(0.20)	(1.98)	(0.21)
Probability in percent ($\overline{p}$)		7.16		9.64	
N=15 ; $N\times T$=249					
Panel C: Currency Crises (baseline sample: country FE and years 1990–2011)					
Reserves	46.63	-4.66***	-0.27***	-8.90***	-0.41***
over External Debt		(1.88)	(0.05)	(1.56)	(0.07)
Net Foreign Assets	11.45	-0.65	0.06	-0.84	-0.08
over GDP		(1.10)	(0.11)	(1.76)	(0.17)
Probability in percent ($\overline{p}$)		5.05		10.35	
N=18 ; $N\times T$=294					

Note: *, **, and *** denote significance at the 10, 5, and 1 percent level. $\partial p/\partial x$ is the marginal effect in percentage at "tranquil" sample mean. δp is the effect in percentage for an increase of one standard deviation in x at the "tranquil" sample mean. $s.d.(x)$ is the unconditional standard deviation of x over "tranquil" times. $\overline{p}$ is the probability of a crisis at the sample mean. Robust standard errors in parentheses are computed using the delta-method. The estimation sample is an unbalanced panel that spans 23 emerging countries between 1990 and 2011. The data stops in 2011 as the updated series by Lane and Milesi-Ferretti (2007) stop in 2011. Due to the use of country fixed effects, countries without a given crisis are not in the logit estimation sample for that type of crisis. Currency, banking, and default crises dates follow Gourinchas and Obstfeld (2012) and are listed in the data appendix. Sample means are higher in Panels B and C as China is part of the estimation sample. The results are similar without China as the current estimation features country fixed effects.

Table A.7: Sensitivity Analysis

(baseline values in parentheses)	1992–1996	1997–2001	2002–2006
$\lambda = 0.5$ (0.6)			
Reserves over External Debt Liabilities	0.21	0.41	0.43
Sudden Stop Probabilities (percent)	0.45	1.67	0.69
$\lambda = 0.7$ (0.6)			
Reserves over External Debt Liabilities	0.17	0.33	0.35
Sudden Stop Probabilities (percent)	0.31	1.21	0.30
$A = 1.1$ (1.2)			
Reserves over External Debt Liabilities	0.24	0.39	0.40
Sudden Stop Probabilities (percent)	0.33	1.60	0.61
$A = 1.3$ (1.2)			
Reserves over External Debt Liabilities	0.17	0.38	0.40
Sudden Stop Probabilities (percent)	0.27	1.08	0.26
$\sigma_H = 0.152$ (0.175)			
Reserves over External Debt Liabilities	0.20	0.35	0.36
Sudden Stop Probabilities (percent)	0.29	1.25	0.47
$\sigma_H = 0.199$ (0.175)			
Reserves over External Debt Liabilities	0.20	0.41	0.43
Sudden Stop Probabilities (percent)	0.29	1.41	0.36
baseline calibration			
Reserves over External Debt Liabilities	0.20	0.39	0.40
Sudden Stop Probabilities (percent)	0.29	1.27	0.37

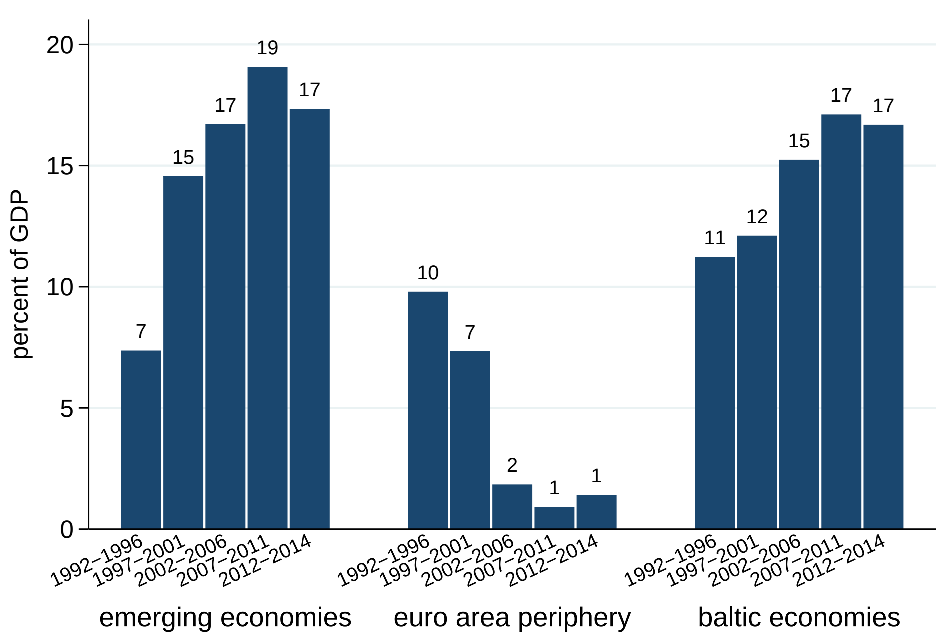
Note: The sensitivity analysis reports data generated by the model in which the parameters of the model have been set to the baseline calibration except for the changes to λ, A, and σ_H respectively.

Table A.8: Panel Logit Estimation on Simulated Data

	S.D.	Crisis in 1–2 years δp	$\frac{\partial p}{\partial x}$	Crisis 1-3 years δp	$\frac{\partial p}{\partial x}$
Panel A: Sudden Stops (one region model - simulated data with country FE)					
Reserves	9.41	-1.43***	-0.18***	-2.43	-0.31***
over External Debt		(0.31)	(0.05)	(7.52)	(0.06)
Probability in percent ($\overline{p}$)		4.86		7.29	
N=23 ; $N \times T \times S$=2738					
Panel B: Sudden Stops (three region model - simulated data with country FE)					
Reserves	8.96	-2.36***	-0.31***	-3.41***	-0.44***
over External Debt		(0.04)	(0.06)	(7.52)	(0.08)
Probability in percent ($\overline{p}$)		7.80		11.83	
N=24 ; $N \times T \times S$=2740					

Note: *, **, and *** denote significance at the 10, 5, and 1 percent level. $\partial p/\partial x$ is the marginal effect in percentage at "tranquil" sample mean. δp is the effect in percentage for an increase of one standard deviation in x at the "tranquil" sample mean. $s.d.(x)$ is the unconditional standard deviation of x over "tranquil" times. $\overline{p}$ is the probability of a crisis at the sample mean. Robust standard errors in parentheses are computed using the delta-method. The estimation sample is an unbalanced panel that spans 23 countries between 1992 and 2006 in 10 simulations. For the three region model, we use 24 countries. The estimation also include simulation fixed effects. The results are similar with more simulations. Naturally, the model features much less dispersion in reserves – and hence fewer crises – compared to the data.

Figure A.10: Foreign Reserves relative to GDP



B. Proofs

B.1. Proof of Proposition 1

We proceed in nine steps.

Step 1: *Interest rates satisfy*

$$r_S^* < 0 < r_N^* \tag{B.1}$$

$1 + r_N^* \geq 1$ follows from equation (7). Equation (2) and $\lambda < 1$ imply that $(R_1 + \lambda K)/D < 1$. Since $\theta = 1$, equation (3) implies $1 + r_S^* = (R_1 + \lambda K)/D$. Hence $r_S^* < 0$.

Step 2: *If* $\psi^*(\varphi) = 1$, *then*

$$\begin{aligned} L^*(\varphi) &= K^* && \text{(B.2)} \\ R_2^*(\varphi) &= 0 && \text{(B.3)} \\ C^*(\varphi) &= 0 && \text{(B.4)} \end{aligned}$$

By definition, if $\psi^*(\varphi) = 1$, then $P_1^*(\varphi) = \left(1 + r_S^*\right) D$. From step 1, we have that $r_S^* = (R_1 + \lambda K)/D$. Equations (4) and (6) imply equations (B.2) and (B.3). Then equation (B.4) follows from equations (5) and (6) .

Step 3: *If* $\psi^*(\varphi) = \varphi$, *then*

$$\begin{aligned} L^*(\varphi) &= 0 && \text{(B.5)} \\ R_2^*(\varphi) &= R_1^* - \varphi D && \text{(B.6)} \\ C^*(\varphi) &= AK^* + R_2^*(\varphi) - (1-\varphi)(1 + r_N^*) D && \text{(B.7)} \end{aligned}$$

By definition, if $\psi^*(\varphi) = \varphi$, then $P_1^*(\varphi) = D$ and $P_2^*(\varphi) = (1 + r_N^*) D$. Suppose for contradiction that $L^*(\varphi) = K$. Then equation (4) implies $R_2^*(\varphi) = R_1^* + \lambda K^* - \varphi D$. Then we have that

$$\begin{aligned} C^*(\varphi) &= R_1^* + \lambda K^* - \varphi D - (1-\varphi)(1 + r_N^*) D \\ &\leq R_1^* + \lambda K^* - D \\ &< 0 \end{aligned}$$

where the first equality comes from equation (5), the second inequality comes from $1 + r_N^* \geq 1$, and the third inequality comes from (2) and $\lambda < 1$. This violates equation (6). Hence equation (B.5) holds. Then equation (B.6) follows from equation (4), and equation (B.7) follows from (5).

Step 4: *If* $\psi^*(\varphi_1) = \varphi_1 < \varphi_2 = \psi^*(\varphi_2)$, *then*

$$R_2^*(\varphi_1) > R_2^*(\varphi_2) \tag{B.8}$$

$$C^*(\varphi_1) < C^*(\varphi_2) \tag{B.9}$$

Equation (B.6) implies that $R_2^*(\varphi_1) = R_1^* - \varphi_1 D > R_1^* - \varphi_2 D = R_2^*(\varphi_2)$. Similarly, step 3 implies that

$$\begin{aligned} C^*(\varphi_1) &= AK^* + R_1^* - \varphi_1 D - (1-\varphi_1)(1+r_N^*)D \\ &< AK^* + R_1^* - \varphi_2 D - (1-\varphi_2)(1+r_N^*)D \\ &= C^*(\varphi_2). \end{aligned}$$

Step 5: *Sudden stop policy satisfies*

$$\exists \varphi_S^* \in [0,1] \text{ s.t. } \begin{cases} \psi^*(\varphi) = \varphi & \forall \varphi \in [0, \varphi_S] \\ \psi^*(\varphi) = 1 & \forall \varphi \in (\varphi_S, 1] \end{cases}$$

First, note that $\psi^*(\varphi) \in \{\varphi, 1\}$, which follows from symmetry. Then, suppose, without loss of generality, that the optimal debt contract B^* has $\varphi_1^* < \varphi_2^* < \varphi_3^*$ such that

$$\psi^*(\varphi) = \begin{cases} 1 & \forall \varphi \in (\varphi_1, \varphi_2] \\ \varphi & \forall \varphi \in (\varphi_2, \varphi_3] \end{cases}$$

Then consider an alternative debt contract $\hat{B}$ that is identical to B^* except that $\hat{\psi}(\varphi) = \varphi \ \forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$ for some $\varepsilon > 0$.

From equations (6) and (B.9), we know that $C^*(\varphi_2) > C^*(0) \geq 0$. By continuity, $\hat{C}(\varphi) > 0 \ \forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$ for ε small enough.

In contrast, from step 2, $C^*(\varphi) = 0 \ \forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$. Similarly, from equations (6) and (B.8), we know that $R_2^*(\varphi_2) > R_2^*(\varphi_3 - \varepsilon) \geq 0$. By continuity, $\hat{R}_2(\varphi) > 0 \ \forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$ for ε small enough.

It remains to show that interim individual rationality, equation (7), holds for any lender with $\varphi^i = 0$ when $\forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$. This is obvious since $\hat{P}_1(\varphi) = D < (1+r_N^*)D = \hat{P}_2(\varphi) \ \forall \varphi \in [\varphi_2 - \varepsilon, \varphi_2]$.

The participation constraint, equation (8), obviously holds since $r_N > 0 > r_S$.

Hence $\hat{B}$ is feasible, yet has strictly higher consumption than B^*, which is a contradiction.

Step 6: *Reserves and liquidation policies satisfy*

$$\exists \varphi_R^* \in [0,1] \text{ s.t. } \begin{cases} R_2^*(\varphi) > 0 & \iff \varphi \in [0, \varphi_R) \\ L^*(\varphi) = 0 & \iff \varphi \in [0, \varphi_R] \end{cases}$$

From step 5, we know that

$$\exists \varphi_S^* \in [0,1] \text{ s.t. } \begin{cases} \psi^*(\varphi) = \varphi & \forall \varphi \in [0, \varphi_S] \\ \psi^*(\varphi) = 1 & \forall \varphi \in (\varphi_S, 1] \end{cases}$$

Let $\varphi_R^* = \varphi_S^*$. Then the result follows from steps 2 and 3. It also follows that $\varphi_R^* = R_1^*/D$.

Step 7: *The Optimal Reserves-to-Debt ratio satisfies*

$$\varphi_R^* = 1 - \left[\frac{A-1}{A-\lambda} \left(\frac{\sigma}{\sigma+1} \right) \right]^{\sigma}$$

The cutoff conditions imply that the state-contingent policy and payment functions can be written as:

$$\begin{aligned} L^*(\varphi) &= \begin{cases} 0 & \text{if } \varphi \leq \varphi_R^* \\ K^* & \text{otherwise} \end{cases} \\ R_2^*(\varphi) &= \begin{cases} R_1^* - \varphi D & \text{if } \varphi \leq \varphi_R^* \\ 0 & \text{otherwise} \end{cases} \\ \psi_i^*(\varphi, \varphi_i) &= \begin{cases} 0 & \text{if } \varphi \leq \varphi_R^* \text{ and } \varphi_i = 0 \\ 1 & \text{otherwise} \end{cases} \\ P_1^*(\varphi) &= \begin{cases} D & \text{if } \varphi \leq \varphi_R^* \\ R_1^* + \lambda K^* & \text{otherwise} \end{cases} \\ P_2^*(\varphi) &= \begin{cases} (1+r_N^*)D & \text{if } \varphi \leq \varphi_R^* \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The participation constraint, holding with equality, can be written as

$$(1+r_W) = G(\varphi_R^*) + (1+r_N^*)\left(F(\varphi_R^*) - G(\varphi_R^*)\right) + \left(1 - F(\varphi_R^*)\right)(1+r_S^*) \tag{B.10}$$

where

$$G(x) \equiv \int_0^x \varphi dF(\varphi).$$

Substituting the resource constraints and the condition $\varphi_R = R_1/D$, the problem becomes:

$$\max_{\varphi_R} J(\varphi_R)$$

where

$$J(\varphi_R) = D\int_0^{\varphi_R}[A(1-\varphi_R)+\varphi_R-\varphi-(1-\varphi)(1+r_N(\varphi_R))]\,dF(\varphi)$$

with

$$r_N(\varphi_R) = \frac{(1+r_W)-F(\varphi_R)-(1-F(\varphi_R))(\lambda+(1-\lambda)\varphi_R)}{F(\varphi_R)-G(\varphi_R)}.$$

The first order condition yields

$$[A-(1+r_N(\varphi_R^*))](1-\varphi_R^*)f(\varphi_R^*)-(A-1)F(\varphi_R^*)-r_N'(\varphi_R^*)(F(\varphi_R^*)-G(\varphi_R^*))=0$$

Substituting (B.10) and its derivative, we get

$$(1-\varphi_R^*)f(\varphi_R^*)+1-F(\varphi_R^*)=\frac{A-1}{A-\lambda}$$

Using the bounded Pareto distribution, we get:

$$\varphi_R^* = 1-\left[\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right]^{\sigma}.$$

Step 8: *To verify the equilibrium is feasible, it suffices to show that*

$$C^*(\varphi)\geq 0\;\forall\varphi\in[0,\varphi_R^*).$$

Since $C^*(\varphi)$ is strictly increasing in φ, it suffices to show $C^*(0)\geq 0$.

$$\begin{aligned} C^*(0) &= (A(1-\varphi_R^*)+\varphi_R^*)D+\frac{G(\varphi_R^*)+(1-F(\varphi_R^*))(\lambda+(1-\lambda)\varphi_R^*)-(1+r_W)}{F(\varphi_R^*)-G(\varphi_R^*)}D \\ &= (A-1)\left[\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right]^{\sigma}D-(\sigma+1)\frac{(1-\lambda)\left(\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right)^{\sigma+1}+r_W}{1-\left(\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right)^{\sigma+1}}D \end{aligned}$$

Note that $\lim_{A\to\infty} C^*(0)=+\infty$. Hence $\exists A^*(\lambda,\sigma,r_W)$ such that $\forall A\geq A^*$, $C^*(0)\geq 0$.

Step 9: *Sufficiency condition*

Having derived the optimal reserves by solving $J'(\varphi_R)=0$, we now verify that $J''(\varphi_R)<0$. We do so using the following three lemmas.

Lemma 1.1: Monotonicity of Interest Rate in Reserves
If $1<(\sigma+1)(1-\lambda+r_W)$, *then* $r_N'(\varphi_R)<0$.

Proof. See Online Appendix.

Lemma 1.2: Convexity of Interest Rates

If $1 < (\sigma+1)(1-\lambda+r_W)$*, then* $r_N''(\varphi_R) > 0$.

Proof. See Online Appendix.

Lemma 1.3: Sufficiency of FOC.

$$J''(\varphi_R) < 0$$

Proof. See Online Appendix.

This concludes the proof of Proposition 1. □

B.2. Proof of Proposition 2

(i) From Proposition 1, we know that

$$\varphi_R^* = 1 - \left[\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right]^{\sigma}.$$

Then,

$$\frac{\partial \varphi_R^*}{\partial \sigma} > 0 \quad \Leftrightarrow \quad \underbrace{\log\left[\frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right)\right] + \frac{1}{\sigma+1}}_{h(\sigma)} < 0$$

Since $\lambda < 1 < A$, this is true since $h(\sigma)$ is increasing in σ, $\lim_{\sigma\to+\infty} h(\sigma) = 0^+$, and $\lim_{\sigma\to 0^+} h(\sigma) = -\infty$; which implies that $h(\sigma) < 0$ for all $\sigma > 0$.

(ii) From Corollary 1, we know that

$$\Pr(\psi = 1) = 1 - F(\varphi_R^*) = \frac{A-1}{A-\lambda}\left(\frac{\sigma}{\sigma+1}\right),$$

which is increasing in σ. □

B.3. Proof of Proposition 3

Reserves Shortfall. Before writing the planner's problem, it is useful to derive how many countries have to suffer a crisis for a given level of reserves shortfall. Suppose all countries coordinate to set $\varphi_R^C = (\bar{\varphi} - \varepsilon)$ reserves aside and invest $\bar{K} + \varepsilon D$ where $\bar{\varphi} = E[\varphi]$ and $\bar{\varphi}D + \bar{K} = D$. The interim shortfall is: εD.

Some countries will have to (fully) liquidate to pay $1+r_S(\varepsilon)=\bar{\varphi}+\lambda\bar{K}/D-(1-\lambda)\,\varepsilon$ since their normal interim payments cannot be met. Let us denote $\ell(\varepsilon)$, the measure of countries that face a crisis. We have: $1-\ell(\varepsilon)=F_\sigma(\widehat{\varphi}(\varepsilon))$ where:

$$\bar{\varphi}-\varepsilon=\int_0^{\widehat{\varphi}(\varepsilon)}\varphi dF_\sigma(\varphi)=G_\sigma(\widehat{\varphi}(\varepsilon))\quad\Leftrightarrow\quad\widehat{\varphi}(\varepsilon)=G_\sigma^{-1}(\bar{\varphi}-\varepsilon)$$

So: $\ell(\varepsilon)=1-F_\sigma\left[G_\sigma^{-1}(\bar{\varphi}-\varepsilon)\right]$.

The reserves decision ε determines the probability $\ell(\varepsilon)$ that a country is in a sudden stop. The shortfall limits the interim insurance since the interim debt repayment of some countries, the ones with the largest shocks, cannot be met. We know:

- $\ell(0)=0$ and $\ell(\bar{\varphi})=1$
- $\ell(\varepsilon)$ is strictly increasing in ε
- $G_\sigma(\varphi)=\frac{\sigma}{\sigma+1}\left[1-\left(1-\frac{1}{\sigma}\varphi\right)(1-\varphi)^{\frac{1}{\sigma}}\right]\quad\Rightarrow\quad\ell(\varepsilon)=\left[1-G_\sigma^{-1}(\bar{\varphi}-\varepsilon)\right]^{\frac{1}{\sigma}}$

We now write the planner's problem as choice of reserves shortfall.

Planner's Problem. Noting that the interim decision has been solved above, the planner's problem is:

$$\max_\varepsilon \quad C$$
$$\text{subject to}$$

$$(\bar{\varphi}-\varepsilon)D+(\bar{K}+\varepsilon D)-D\leq 0 \tag{B.11}$$
$$C+\left[\int_0^{\widehat{\varphi}(\varepsilon)}(1-\varphi)\,dF_\sigma(\varphi)\right](1+r_N)D-A(1-\ell(\varepsilon))(\bar{K}+\varepsilon D)\leq 0 \tag{B.12}$$
$$\ell(\varepsilon)(1+r_S(\varepsilon))+\int_0^{\widehat{\varphi}(\varepsilon)}[\varphi+(1-\varphi)(1+r_N)]\,dF_\sigma(\varphi)-(1+r_W)\geq 0 \tag{B.13}$$
$$(1+r_S(\varepsilon))-[(\bar{\varphi}-\varepsilon)D+\lambda(\bar{K}+\varepsilon D)]\geq 0 \tag{B.14}$$
$$C,\ \varepsilon\geq 0 \tag{B.15}$$

Equations (B.11) - (B.15) represent initial resource constraint, final resource constraint, participation constraint, renegotiation proofness, and non-negativity constraint, which are analogous to equations (2), (5), (8), (3) , and (6), respectively. This simplifies to:

$$\max_\varepsilon \quad C$$
$$\text{subject to}$$

$$C+[1-\ell(\varepsilon)-(\bar{\varphi}-\varepsilon)](1+r_N)D-A(1-\ell(\varepsilon))(\bar{K}+\varepsilon D) \leq 0 \tag{B.16}$$

$$\ell(\varepsilon)(1+r_S(\varepsilon))+\int_0^{\widehat{\varphi}(\varepsilon)}[\varphi+(1-\varphi)(1+r_N)]\,dF_\sigma(\varphi)-(1+r_W) \leq 0 \tag{B.17}$$

$$(1+r_S(\varepsilon))-[(\bar{\varphi}-\varepsilon)D+\lambda(\bar{K}+\varepsilon D)] \geq 0 \tag{B.18}$$

$$C,\ \varepsilon \geq 0 \tag{B.19}$$

Equation (B.17) can be written as:[41]

$$\ell(\varepsilon)(1+r_S(\varepsilon))+(\bar{\varphi}-\varepsilon)+(1+r_N)[1-\ell(\varepsilon)-(\bar{\varphi}-\varepsilon)] = (1+r_W) \tag{B.20}$$

Substituting (B.20) into (B.16) yields:

$$C+((1+r_W)-(\bar{\varphi}-\varepsilon)-\ell(\varepsilon)(1+r_S(\varepsilon)))D-A(1-\ell(\varepsilon))(\bar{K}+\varepsilon D)\leq 0$$

The planner's problem can then be written as:

$$\max_{\varepsilon} \quad -\ell(\varepsilon)A\frac{\bar{K}}{D}+A(1-\ell(\varepsilon))\varepsilon-\varepsilon+\ell(\varepsilon)\left[\bar{\varphi}+\lambda\frac{\bar{K}}{D}-(1-\lambda)\varepsilon\right]$$

This is not a linear problem in ε since $\ell(\varepsilon)$ is not linear. However, we know that:

$$\ell(\varepsilon) = 1-F_\sigma\left[G_\sigma^{-1}(\bar{\varphi}-\varepsilon)\right]$$

So,

$$\ell'(\varepsilon)=-F_\sigma'\left[G_\sigma^{-1}(\bar{\varphi}-\varepsilon)\right]\times\left(\frac{1}{G_\sigma'\left[G_\sigma^{-1}(\bar{\varphi}-\varepsilon)\right]}\right)\times(-1)=\frac{1}{\widehat{\varphi}(\varepsilon)}.$$

The F.O.C. w.r.t. ε gives:

$$-\ell'(\varepsilon)A\frac{\bar{K}}{D}-\ell'(\varepsilon)A\varepsilon+A(1-\ell(\varepsilon))-1+\ell'(\varepsilon)\left[\bar{\varphi}+\lambda\frac{\bar{K}}{D}-(1-\lambda)\varepsilon\right]-(1-\lambda)\ell(\varepsilon)\geq 0$$

with equality if $\varepsilon>0$.

Rearranging yields

$$(A+1-\lambda)F_\sigma(\widehat{\varphi}(\varepsilon))-\frac{(A-\lambda)-(A+1-\lambda)(\bar{\varphi}-\varepsilon)}{\widehat{\varphi}(\varepsilon)}-(2-\lambda)\geq 0$$

At $\varepsilon=0$, the L.H.S. of the F.O.C. is $(A+1-\lambda)\bar{\varphi}-(1-\lambda)$, which is positive iff $\sigma>(1-\lambda)/A$.

[41]This is assuming the shortfall is not too high. Otherwise, the consumption would be negative due to the high interest implied by the high reserves shortfall.

Therefore, $\varphi_R^C = \bar{\varphi}$ if and only if $\sigma \leq (1-\lambda)/A$. Otherwise, $\varphi_R^C < \bar{\varphi}$. Obviously, in any case: $\varphi_R^C \leq \bar{\varphi}$.

Finally, given that $\varphi_R^* = 1 - [(A-1)/(A-\lambda)\,\sigma/(\sigma+1)]^{\sigma}$ and $\bar{\varphi} = \frac{\sigma}{\sigma+1}$:

$$\varphi_R^* > \bar{\varphi} \Leftrightarrow \frac{1}{\sigma+1} > \left(\frac{A-1}{A-\lambda}\right)^{\sigma} \left(\frac{\sigma}{1+\sigma}\right)^{\sigma}$$

Since $\lambda < 1 < A$, it is sufficient to show that:

$$\frac{1}{\sigma+1} > \left(\frac{\sigma}{1+\sigma}\right)^{\sigma},$$

which is true for $\sigma \in (0,1)$. $\square$

B.4. Proof of Proposition 4

Given an arbitrary stage-0 allocation $\{\hat{R}_1, \hat{K}\}$ and normal time interest rate $\{\hat{r}_N\}$, we first characterize the thresholds for which the government is solvent. Then, after characterizing the allocations under normal times ($\psi < 1$) and under sudden stop ($\psi = 1$), we characterize the optimal sudden stop region. Finally, we characterize the implied interest rates. Without loss of generality, normalize $D = 1$.

Solvency Region. First, let us denote $d \in [0,1]$ the fraction of reserves $\hat{R}_1$ drawn in the interim. Similarly, let $\ell \in [0,1]$ denote the fraction of capital K liquidated in the interim. The non crisis resource constraints are:

$$\begin{array}{llcl}
[\text{RC0}] & \hat{R}_1 & = & 1 + R_0 - \hat{K} \\
[\text{RC1}] & \varphi & = & d\hat{R}_1 + \lambda \ell \hat{K} \\
[\text{RC2}] & C + R' + (1-\varphi)(1+\hat{r}_N) & = & (1-d)R_1 + A(1-\ell)K
\end{array}$$

with $d \in [0,1]$, $\ell \in [0,1]$, $K \in [0,1]$, $C \geq 0$, and $R' \geq 0$.

Substituting [RC1] into [RC2], any resource feasible liquidation $\ell \in [0,1]$ satisfies:

$$(A-\lambda)\,\ell\hat{K} \leq A\hat{K} + \hat{R}_1 - 1 - (1-\varphi)\,\hat{r}_N$$

Using [RC0], we obtain:

$$(A-\lambda)\,\ell\hat{K} \leq (A-1)\,\hat{K} + R_0 - (1-\varphi)\,\hat{r}_N$$

The level of liquidation which makes the government insolvent in period 2 is

$$\widehat{\ell}(\varphi) = \min\left\{\frac{(A-1)\hat{K}+R_0-\hat{r}_N}{(A-\lambda)\hat{K}} + \frac{\hat{r}_N}{(A-\lambda)\hat{K}}\varphi, 1\right\}$$

Denote:

$$\varphi_{liq} \equiv \min_{[0,1]}\left\{\varphi \text{ s.t. } \widehat{\ell}(\varphi) \geq 0\right\} = \min\left\{\max\left\{-\frac{(A-1)K+R_0-r_N}{r_N}, 0\right\}, 1\right\} \tag{B.21}$$

To further guarantee solvency in time 1, we need: $\varphi \leq \hat{R}_1 + \lambda\widehat{\ell}(\varphi)\hat{K}$. Denote:

$$[\varphi_{lo}, \varphi_{up}] \equiv \left\{\varphi \in [0,1] \text{ s.t. } \varphi \leq \hat{R}_1 + \lambda\widehat{\ell}(\varphi)\hat{K}\right\} \tag{B.22}$$

The government is solvent over

$$[\varphi_{min}, \varphi_{max}] \equiv \left[\varphi_{liq}, 1\right] \cap [\varphi_{lo}, \varphi_{up}] \tag{B.23}$$

Denote $\varphi_R = \min\left\{1, \hat{R}_1\right\}$. Note that $\varphi_R \leq \varphi_{up}$ and $\varphi_R \leq \varphi_{max}$.

Normal Time Net Output. During normal times, when $\psi < 1$, net output is given by:

$$Y(\varphi) = A\hat{K} + \hat{R}_1 - 1 - \hat{r}_N(1-\varphi) - (A-\lambda)\ell\hat{K}$$

Since $Y(\varphi)$ is decreasing in ℓ, and thereby increasing in d, it is optimal to pay first with reserves. Hence, if $\varphi \leq \hat{R}_1$ then

$$(d,\ell) = \left(\frac{\varphi}{\hat{R}_1}, 0\right) \tag{B.24}$$

$$Y(\varphi) = A\hat{K} + \hat{R}_1 - 1 - \hat{r}_N(1-\varphi) \tag{B.25}$$

Otherwise, if $\varphi > \hat{R}_1$ then

$$(d,\ell) = \left(1, \frac{\varphi - \hat{R}_1}{\lambda\hat{K}}\right) \tag{B.26}$$

$$Y(\varphi) = A\hat{K} + \hat{R}_1 - 1 - \hat{r}_N(1-\varphi) - (A-\lambda)\left(\frac{\varphi - \hat{R}_1}{\lambda}\right) \tag{B.27}$$

Overall, we have:

$$Y(\varphi) = A\hat{K} + \hat{R}_1 - 1 - \hat{r}_N(1-\varphi) - (A-\lambda)\left(\frac{\varphi - \hat{R}_1}{\lambda}\right)^+ \tag{B.28}$$

Sudden Stop Net Output. During a sudden stop, when $\psi = 1$, net output is

$$Y_{SS} = A\hat{K} + \hat{R}_1 - 1 - r_S - (A - \lambda)\,\ell\hat{K}$$

where

$$[\text{NP}] \quad 1 + r_S = \min\left\{1, \theta\frac{\hat{R}_1 + \lambda\hat{K}}{D}\right\}.$$

Since Y_{SS} is decreasing in ℓ, it is optimal to pay first with reserves. Hence, if $\hat{R}_1 \geq (1 + r_S)$ then

$$(d, \ell) = \left(\frac{1 + r_S}{\hat{R}_1}, 0\right) \tag{B.29}$$

$$Y_{SS} = A\hat{K} + \hat{R}_1 - 1 - r_S \tag{B.30}$$

Otherwise, if $\hat{R}_1 < (1 + r_S)$ then

$$(d, \ell) = \left(1, \frac{1 + r_S - \hat{R}_1}{\lambda\hat{K}}\right) \tag{B.31}$$

$$Y_{SS} = A\hat{K} + \hat{R}_1 - 1 - r_S - (A - \lambda)\left(\frac{1 + r_S - \hat{R}_1}{\lambda}\right) \tag{B.32}$$

Overall, we have:

$$Y_{SS} = A\hat{K} + \hat{R}_1 - 1 - r_S - (A - \lambda)\left(\frac{1 + r_S - \hat{R}_1}{\lambda}\right)^+ \tag{B.33}$$

Sudden Stop Region. A sudden stop is optimal when the net output in normal times is less than the sudden stop net output. That is:

$$\varphi\hat{r}_N - (A - \lambda)\left(\frac{\varphi - \hat{R}_1}{\lambda}\right)^+ \leq \hat{r}_N - r_S - (A - \lambda)\left(\frac{1 + r_S - \hat{R}_1}{\lambda}\right)^+$$

First, consider $\varphi \in \left[0, \hat{R}_1\right] \cap [\varphi_{min}, \varphi_{max}]$. Then $Y(\varphi) \leq Y_{SS}$ if and only if

$$\varphi \leq \frac{(\hat{r}_N - r_S) - \frac{A - \lambda}{\lambda}\left(1 + r_S - \hat{R}_1\right)^+}{\hat{r}_N}$$

Second, consider $\varphi \in \left[\hat{R}_1, 1\right] \cap [\varphi_{min}, \varphi_{max}]$. Then $Y(\varphi) \leq Y_{SS}$ if and only if

$$\varphi \geq \frac{\frac{A-\lambda}{\lambda}\hat{R}_1 + \frac{A-\lambda}{\lambda}\left(1 + r_S - \hat{R}_1\right)^+ - (\hat{r}_N - r_S)}{\frac{A-\lambda}{\lambda} - \hat{r}_N}$$

Denote:

$$\underline{\varphi} \equiv \min\left\{\hat{R}_1, \frac{(\hat{r}_N - r_S) - \frac{A-\lambda}{\lambda}\left(1 + r_S - \hat{R}_1\right)^+}{\hat{r}_N}\right\} \tag{B.34}$$

$$\overline{\varphi} \equiv \max\left\{\hat{R}_1, \frac{\frac{A-\lambda}{\lambda}\hat{R}_1 + \frac{A-\lambda}{\lambda}\left(1 + r_S - \hat{R}_1\right)^+ - (\hat{r}_N - r_S)}{\frac{A-\lambda}{\lambda} - \hat{r}_N}\right\} \tag{B.35}$$

$$\varphi_N \equiv \max\left\{\varphi_{min}, \underline{\varphi}\right\} \tag{B.36}$$

$$\varphi_S \equiv \min\left\{\varphi_{max}, \overline{\varphi}\right\} \tag{B.37}$$

Note that $\overline{\varphi} \geq \varphi_R$ and $\varphi_{max} \geq \varphi_R$. Hence $\varphi_S \geq \varphi_R$. Therefore, by equation B.24, for any $\varphi \in (\varphi_N, \varphi_R)$, liquidation is zero.

Implied Interest Rate. Let $\mathcal{N} = [\varphi_N, \varphi_S]$. The normal time interest is then determined from

$$[\text{PC}]\ (1 + r_W) = G_\rho(\mathcal{N}) + (1 + r_N)\left[F_\rho(\mathcal{N}) - G_\rho(\mathcal{N})\right] + (1 + r_S)\left[1 - F_\rho(\mathcal{N})\right]$$

where $F_\rho(\varphi) = \rho f_{\sigma_L} + (1-\rho) f_{\sigma_H}$ and $G_\rho(\varphi) \equiv \int_0^\varphi x dF_\rho(x)$.

Since this characterization holds for any arbitrary stage-0 allocation, it also holds for the optimal stage-0 allocation. □

B.5. Proposition 5. Mutual Insurance with Correlated Shocks

Let $\gamma \in [0,1]$. Consider a measure of *ex ante* identical countries with randomly assigned identity $j \in [0,1]$ subject to liquidity shocks $\left\{\varphi^j\right\}_{j\in[0,1]}$. Correlation is modeled as follows:

$$\begin{cases} \varphi^j = \varphi(\gamma) & \forall j \in (0,\gamma) \quad \text{where } \varphi(\gamma) \sim F_\sigma \\ \varphi^j = \varphi(j) & \forall j \in (\gamma,1) \quad \text{where } \varphi(j) \sim F_\sigma \end{cases}$$

In other words, the liquidity shocks are perfectly correlated across the countries in $(0,\gamma)$ and i.i.d across the countries in $(\gamma,1)$. Hence, γ reflects the correlation of shocks across countries.
Under mutual insurance, the optimal reserves held by each country is

$$\gamma\varphi_R^* + (1-\gamma)\varphi_R^C \in \left[\varphi_R^*, \varphi_R^C\right]$$

Proof. Suppose that there are two liquidity insurance agreements setup in the interim, after the identity j is known. In particular, there is one insurance for $(0,\gamma)$ and another for $(\gamma,1)$.
From Proposition 1, we know that the optimal reserve for $(0,\gamma)$ is φ_R^* since the countries are identical *ex post* and cannot insure each other.
Similarly, from Proposition 3, the optimal reserve for $(\gamma,1)$ is φ_R^C.

Could the two pools cross-insure by distorting these reserve allocations? No.
Moreover, *ex post*, no other insurance groups can be formed. Hence, *ex ante*, the optimal reserve allocation chosen is:

$$\gamma\varphi_R^* + (1-\gamma)\,\varphi_R^C \in \left[\varphi_R^*, \varphi_R^C\right].$$

This concludes the proof of proposition 5. □

B.6. Proposition 6. Conditional Distribution Function of Posterior Beliefs

$$\Pr\left[\rho' \leq x \mid \varphi^j,\, \rho\right] \;=\; \rho\left[1 - A_{N-1}\left(m\left(x,\varphi^j\right) \mid \sigma_L\right)\right] + (1-\rho)\left[1 - A_{N-1}\left(m\left(x,\varphi^j\right) \mid \sigma_H\right)\right]$$

where

$$m\left(x,\varphi^j\right) \;\equiv\; -\log\left(1-\varphi^j\right) + \left(\sigma_L^{-1} - \sigma_H^{-1}\right)\log\left[\left(x^{-1}-1\right) \times \left(\frac{\rho}{1-\rho}\right) \times \left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)^N\right]$$

$$A_N\left(y \mid \sigma\right) \;\equiv\; \exp\left(\frac{y}{\sigma}\right)\sum_{n=1}^{N}(-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-1}}y^{n-1}$$

Proof. See Online Appendix. □

Online Appendix

C. Computational Appendix

The government solves

$$W(R_0;\rho) = \max_{B\in\Gamma(R_0;\rho)} \mathbf{E}_{\vec{\varphi}|\rho}\left[C(R_0,\vec{\varphi};\rho) + \beta W\left(R_0'(R_0,\vec{\varphi};\rho);\rho'(\rho,\vec{\varphi})\right)\right]$$

This problem involves the N-dimensional vector $\vec{\varphi}$ as an interim contingency indexing each of the interim decisions $\{\psi, L, R_0'\}$. Therefore, the size of the vector of shocks $\vec{\varphi}$ substantially increases the computational burden of the problem. Below, we replace the interim state $\vec{\varphi}\in[0,1]^N$ with the sufficient representation $\{\varphi^j, \rho'(\rho,\vec{\varphi})\}\in[0,1]^2$.

$$\begin{aligned} W(R_0;\rho) &= \max_{K\in\widehat{\Gamma}(R_0;\rho)} \int \left\{ \max_{R_0'\in[0,R_{max}(\varphi;\rho,R_0,K)]} \left[u\Big(C\left(R_0',\varphi;\rho,R_0,K\right)\Big) \right.\right. \\ &\qquad \left.\left. + \beta \int W\left(R_0';\rho'\right) g(\rho';\varphi,\rho)\mathrm{d}\rho' \right] \right\} h^1_\rho(\varphi)\mathrm{d}\varphi \end{aligned}$$

In this representation, h^1_ρ is the perceived distribution of local shocks given a prior ρ. Hence, $h^1_\rho = \rho f_{\sigma_L} + (1-\rho) f_{\sigma_H}$. Most importantly, we derive an analytical expression for the conditional distribution g of posterior beliefs ρ' (see Proposition 6 in Appendix B.6).

C.1. Algorithm: Feasibility

We derive the feasible contract for each tuple of belief, incoming reserves and capital choice. The main step in deriving the feasible contracts is finding the interest rate satisfying the participation constraint, given the endogenous crisis regions. We derive the constraint set $\left\{(R_0,\rho,K) \mid K\in\widehat{\Gamma}(R_0;\rho)\right\}$ once and for all. A regular value function iteration is then computed.

For each ρ, R_0 and $K\in[0,1]$

1. Derive $R_1 = R_0 + 1 - K$ using [RC0]
2. Derive $\{r_N, r_S, \varphi_N, \varphi_S\}$ using Proposition 4, and r_S from [NPC]
3. Set feasibility indicator to 0 if there is no solution to the participation constraint.
4. Store $R_1, r_N, r_S, \varphi_N, \varphi_S$

C.2. Algorithm: Value Function Iteration

We can further rewrite the problem as:

$$W(R_0,\rho) = \max_{K\in\widehat{\Gamma}(R_0;\rho)} J(K,R_0,\rho)$$

where:

$$J(K,R_0,\rho) \quad = \quad \iint V\left(\varphi,\rho';K,R_0,\rho\right) g(\rho';\varphi,\rho) h^1_\rho(\varphi)\mathrm{d}\rho'\mathrm{d}\varphi$$

Given the constraint set, a simple value function iteration on a discrete state space is used to compute the optimal policies. Below is a sketch of the solution method. Guess a value function $W_{\text{old}}(R_0;\rho) \equiv W(R_0;\rho)$. Then, iterate on value function:

1. For each ρ, R_0, K
   (a) Skip if (R_o, ρ, K) is not in the constraint set.
   (b) Retrieve the policies $R_1, r_H, r_L, \underline{\varphi}, \overline{\varphi}$ derived from feasibility
      i. For each crisis state $\varphi \notin [\underline{\varphi}, \overline{\varphi}]$,
         - For each possible posterior $\rho' \in [0,1]$
           - Initialize crisis value function $V_{candidate}(R';\varphi,\rho') = 0$
           - Compute output Y_{SS} from (B.33)
           - For each candidate savings $R' \in [0, Y_{SS}]$, compute

$$V_{candidate}(R';\varphi,\rho') = u(Y_{SS} - R') + \beta W(R';\rho')$$

           - Store $V(\varphi,\rho') = \max_{R'} V_{candidate}(R';\varphi,\rho')$
      ii. For each non-crisis state $\varphi \in [\underline{\varphi}, \overline{\varphi}]$
         - For each possible posterior $\rho' \in [0,1]$
           - Initialize non crisis value function $V_{candidate}(R';\varphi,\rho') = 0$
           - Compute output $Y(\varphi)$ from (B.28)
           - For each candidate savings $R' \in [0, Y(\varphi)]$, compute

$$V_{candidate}(R';\varphi,\rho') = u(Y(\varphi) - R') + \beta W(R';\rho')$$

           - Store $V(\varphi,\rho') = \max_{R'} V_{candidate}(R';\varphi,\rho')$
      iii. Compute expected value

$$J(K) = \int_0^1 \int_0^1 V(\varphi,\rho')\, g(\rho';\varphi,\rho)\, h^1_\rho(\varphi)\mathrm{d}\rho'\mathrm{d}\varphi$$

   - Find optimal stage-0 allocation K by solving

$$W_{\text{new}}(R_0;\rho) = \max_K J(K)$$

$$K(R_0;\rho) = \arg\max_K J(K)$$

2. Update value function guess

$$W_{\text{old}} = \omega W_{\text{old}} + (1-\omega) W_{\text{new}}$$

3. Repeat (1) until convergence

$$||W_{\text{new}} - W_{\text{old}}|| < \varepsilon$$

We use 150 points for liquidity shocks, 5 points for incoming beliefs, 10 points for posterior beliefs, 40 points for incoming saved reserves, 60 points for capital, and 20 points for outgoing saved reserves on a state-contingent grid. We use linear interpolation when the optimal decision does not fall on a grid point. The results are robust to more grid points and alternative interpolation methods. The FORTRAN code used is available online.

C.3. Simulation and Calibration

For a given parameter configuration, we simulate the quarterly model a large number of times, each time for N_{EME} countries and $T = 1992Q1 \ldots 2006Q4$. Given the optimal policy functions and the updating rule for posterior beliefs, we simulate the model iterating on the rules to obtain the endogenous paths for reserves, beliefs, and sudden stops. In order to let countries fully learn about the initial regime, we first let each model simulation run many quarters leading up to $T = 1992Q1$. The calibration is accomplished by choosing the parameter configuration that minimizes the equally weighted mean-squared errors.

For each $\beta, \sigma_L, \sigma_H, \theta$ and for each $s = 1 : N_S$ simulations:

- Draw $N_{EME} * T_{END}$ random draws in $[0,1]$ such that $\sigma = \sigma_L$ for $t = 1 : 1996Q4$ and $\sigma = \sigma_H$ for $t = 1997Q1 : 2006Q4$
- Apply iteratively policy functions and belief updating rules
- Save average R_1 from $t = 1992Q1 : 1996Q4$ across time and countries
- Save average R_1 from $t = 2002Q1 : 2006Q4$ across time and countries
- Save average haircut $-r_S$ during sudden stops
- Save sudden stop frequency across $t = 1992Q1 : 1996Q4$ and $t = 2002Q1 : 2006Q4$

D. Additional Proofs

D.1. Lemma 1.1: Monotonicity of Interest Rate in Reserves

If $1 < (\sigma+1)(1-\lambda+r_W)$, *then* $r_N'(\varphi_R) < 0$.
Proof.

$$\begin{aligned} r_N'(\varphi_R) &= \frac{-f(\varphi_R)+f(\varphi_R)(\lambda+(1-\lambda)\varphi_R)-[1-F(\varphi_R)](1-\lambda)}{(F(\varphi_R)-G(\varphi_R))} \\ &\quad -\frac{1+r_W-F(\varphi_R)-[1-F(\varphi_R)](\lambda+(1-\lambda)\varphi_R)}{(F(\varphi_R)-G(\varphi_R))^2}(1-\varphi_R)f(\varphi_R) \end{aligned}$$

Using the Pareto Distribution properties, we have:

$$\begin{aligned} r_N'(\varphi_R) &= \frac{(1-\varphi_R)f(\varphi_R)}{(F(\varphi_R)-G(\varphi_R))^2}\Bigg\{[\lambda-(1-\lambda)\sigma](F(\varphi_R)-G(\varphi_R)) \\ &\qquad -[1-F(\varphi_R)](1-\lambda)(1-\varphi_R)-r_W\Bigg\} \end{aligned}$$

We also know that:

$$\begin{aligned} F(\varphi_R)-G(\varphi_R) &= \frac{1}{\sigma+1}-\frac{1}{\sigma+1}(1-\varphi_R)^{\frac{1}{\sigma}+1} \\ [1-F(\varphi_R)](1-\varphi_R) &= (1-\varphi_R)^{\frac{1}{\sigma}+1} \\ &= (\sigma+1)\left[-(F(\varphi_R)-G(\varphi_R))+\frac{1}{\sigma+1}\right] \end{aligned}$$

Therefore:

$$r_N'(\varphi_R) = \frac{(1-\varphi_R)f(\varphi_R)}{(F(\varphi_R)-G(\varphi_R))^2}\left\{F(\varphi_R)-G(\varphi_R)-(1-\lambda)-r_W\right\}$$

Finally, ☺

$$\begin{aligned} r_N'(\varphi_R) &< 0 \\ &\Leftrightarrow \\ 0 &< F(\varphi_R)-G(\varphi_R)-(1-\lambda)-r_W \\ &\Leftrightarrow \\ 0 &< \frac{1}{\sigma+1}-\frac{1}{\sigma+1}(1-\varphi_R)^{\frac{1}{\sigma}+1}-(1-\lambda)-r_W, \end{aligned}$$

which holds if $1 < (\sigma+1)(1-\lambda+r_W)$. □

D.2. Lemma 1.2: Convexity of Interest Rates

If $1 < (\sigma+1)(1-\lambda+r_W)$, *then* $r_N''(\varphi_R) > 0$.

Proof. From above lemma, we have:

$$r_N'(\varphi_R) = \frac{(1-\varphi_R)f(\varphi_R)}{F(\varphi_R)-G(\varphi_R)}\left[1-\frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)}\right]$$

Then,

$$\begin{aligned} r_N''(\varphi_R) &= \frac{-f(\varphi_R)+(1-\varphi_R)f'(\varphi_R)}{F(\varphi_R)-G(\varphi_R)}\left[1-\frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)}\right] \\ &\quad -\frac{(1-\varphi_R)^2 f(\varphi_R)^2}{(F(\varphi_R)-G(\varphi_R))^2}\left[1-\frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)}\right] \\ &\quad +\frac{(1-\varphi_R)f(\varphi_R)}{F(\varphi_R)-G(\varphi_R)}\frac{1-\lambda+r_W}{(F(\varphi_R)-G(\varphi_R))^2}(1-\varphi_R)f(\varphi_R) \\ &= \frac{-f(\varphi_R)+(1-\varphi_R)f'(\varphi_R)}{F(\varphi_R)-G(\varphi_R)}\left[1-\frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)}\right] \\ &\quad -\frac{(1-\varphi_R)^2 f(\varphi_R)^2}{(F(\varphi_R)-G(\varphi_R))^2}\left[1-2\frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)}\right] \end{aligned}$$

Hence:

$$\begin{aligned} r_N''(\varphi_R) &> 0 \\ &\Leftarrow \\ 1 &< \frac{1-\lambda+r_W}{F(\varphi_R)-G(\varphi_R)} \end{aligned}$$

which is the same condition needed for $r_N'(\varphi_R) < 0$. This lemma guarantees that the constraint set is convex. □

D.3. Lemma 1.3: Sufficiency of FOC

$$J''(\varphi_R) < 0$$

Proof. We know that

$$J'(\varphi_R) = [A-(1+r_N(\varphi_R))](1-\varphi_R)f(\varphi_R) - (A-1)F(\varphi_R) - r_N'(\varphi_R)(F(\varphi_R)-G(\varphi_R))$$

Then

$$
\begin{aligned}
J''(\varphi_R) &= -r'_N(\varphi_R)(1-\varphi_R)f(\varphi_R) \\
&\quad + [A-(1+r_N(\varphi_R))]\left[-f(\varphi_R)+(1-\varphi_R)f'(\varphi_R)\right] \\
&\quad -(A-1)f(\varphi_R) \\
&\quad -r''_N(\varphi_R)(\varphi_R)\left(F(\varphi_R)-G(\varphi_R)\right) \\
&\quad -r'_N(\varphi_R)(1-\varphi_R)f(\varphi_R) \\
&= -\left[A-1+A-(1+r_N(\varphi_R))+2r'_N(\varphi_R)(1-\varphi_R)\right]f(\varphi_R) \\
&\quad + [A-(1+r_N(\varphi_R))](1-\varphi_R)f'(\varphi_R) \\
&\quad -r''_N(\varphi_R)\left(F(\varphi_R)-G(\varphi_R)\right)
\end{aligned}
$$

It suffices to show

$$
2(A-1)-r_N(\varphi_R)+2r'_N(\varphi_R)(1-\varphi_R)>0
$$

$$
\Leftrightarrow
$$

$$
\begin{gathered}
2(A-1)\left(F(\varphi_R)-G(\varphi_R)\right)^2 \\
-\{r_W+[1-F(\varphi_R)](1-\lambda)(1-\varphi_R)\}\left(F(\varphi_R)-G(\varphi_R)\right) \\
+2(1-\varphi_R)^2 f(\varphi_R)\left[F(\varphi_R)-G(\varphi_R)-(1-\lambda+r_W)\right]>0
\end{gathered}
$$

We know that

$$
\begin{aligned}
\lim_{A\to\infty}(1-\varphi_R^*) &= \lim_{A\to\infty}\left(\frac{A-1}{A-\lambda}\frac{\sigma}{\sigma+1}\right)^{\sigma} = \left(\frac{\sigma}{\sigma+1}\right)^{\sigma} \\
\lim_{A\to\infty} J''(\varphi_R^*)) &= -\infty
\end{aligned}
$$

Hence $\exists A^*(\lambda,\sigma,r_W)$ such that $\forall A \geq A^*$, $J''(\varphi_R)<0$. $\square$

D.4. Proof of Proposition 6

We know that by Bayes' rule:

$$\begin{aligned}\rho'(\rho,\vec{\varphi}) &= \left[1+\left(\frac{1-\rho}{\rho}\right)\times\frac{f^N_{\sigma_H}(\vec{\varphi})}{f^N_{\sigma_L}(\vec{\varphi})}\right]^{-1} \\ &= \left[1+\left(\frac{1-\rho}{\rho}\right)\times\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\times\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]^{\sigma_H^{-1}-\sigma_L^{-1}}\right]^{-1}\end{aligned}$$

Let $\bar{x}(\rho)\equiv\left[1+\left(\frac{1-\rho}{\rho}\right)\times\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\right]^{-1}$. Since $\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]\in[0,1]$, we know that $\rho'(\rho,\vec{\varphi})\in(0,\ \bar{x}(\rho))$.

Then, $\forall x\in(0,\bar{x}(\rho))$:

$$\begin{aligned}\Pr\left(\rho'\le x\right) &= \Pr\left\{\left[1+\left(\frac{1-\rho}{\rho}\right)\times\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\times\left(\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]\right)^{\sigma_H^{-1}-\sigma_L^{-1}}\right]^{-1}\le x\right\} \\ &= \Pr\left\{\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]\le\left[\left(x^{-1}-1\right)\times\left(\frac{\rho}{1-\rho}\right)\times\left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)^N\right]^{\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}}}\right\}\end{aligned}$$

Let us denote:

$$m(x,\rho)\equiv\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}}\left[\log\left(x^{-1}-1\right)+\log\left(\frac{\rho}{1-\rho}\right)+N\log\left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)\right]$$

Note that: $x\in(0,\bar{x})\Leftrightarrow m(x,\rho)\in(-\infty,0)\Leftrightarrow\exp m(x,\rho)\in(0,1)$.

We therefore derive:

$$\begin{aligned}\Pr\left(\rho'\le x\right) &= \Pr\left[\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]\le\exp m(x,\rho)\right] \\ &= \rho\Pr\left[\sum_{j=1}^{N}\log\left[1-\vec{\varphi}(j)\right]\le m(x,\rho)\mid\sigma_L\right]+ \\ &\quad (1-\rho)\Pr\left[\sum_{j=1}^{N}\log\left[1-\vec{\varphi}(j)\right]\le m(x,\rho)\mid\sigma_H\right]\end{aligned}$$

To characterize the distribution of the posterior, we establish the theorem below.

Theorem 6.1

$$\Pr\left[\sum_{j=1}^{N}\log\left[1-\vec{\varphi}(j)\right]\leq y\mid\sigma\right]=\frac{1}{\sigma}\times a_N(y)\times\exp\left(\frac{y}{\sigma}\right)\equiv A_N(y)$$

where:

$$\begin{aligned}a_1(y) &= \sigma\\ a_{n+1}(y) &= a_n(y)+\int_y^0\left[a_n'(z)+\frac{1}{\sigma}a_n(z)\right]\mathrm{d}z\end{aligned}$$

Proof by recursion. Let us denote: $Y_j\equiv\log\left(1-\varphi_j\right)$ where $\varphi_j\sim F_\sigma$. Define $Z_N\equiv\sum_{j=1}^{N}Y_j$. Let us derive by recursion, the c.d.f. A_N of Z_N. This will prove the first part of the theorem.

Case where $n=1$ First, we know that: $\Pr\left(1-\varphi\leq x\mid\sigma\right)=x^{\sigma^{-1}}$.
So: $\Pr\left(\log\left(1-\varphi\right)\leq y\mid\sigma\right)=\exp\left(\frac{y}{\sigma}\right)$.
Therefore, by observation, $a_1(y)=\sigma$ and $A_1(y)=\exp\left(\frac{y}{\sigma}\right)$.

Recursion Let us now prove that the property holds at $n+1$, assuming it holds for n.
In other words, let us assume that:

$$\Pr\left(Z_n\leq y\mid\sigma\right)=\frac{1}{\sigma}\times a_n(y)\times\exp\left(\frac{y}{\sigma}\right)\equiv A_n(y)$$

We have $\forall\, y<0$:

$$\begin{aligned}\Pr\left(Z_{n+1}\leq y\mid\sigma\right) &= \Pr\left(\sum_{j=1}^{n+1}Y_j\leq y\mid\sigma\right)\\ &= \Pr\left(Z_n+Y_1\leq y\mid\sigma\right)\\ &= \Pr\left(Y_1\leq y-Z_n\mid\sigma\right)\\ &= \int_{-\infty}^{0}\Pr\left(Y_1\leq y-z_n\mid\sigma\right)\mathrm{d}A_n(z_n)\\ &= \int_{-\infty}^{y}\mathrm{d}A_n(z_n)+\int_y^0\Pr\left(Y_1\leq y-z_n\mid\sigma\right)\mathrm{d}A_n(z_n)\\ &= A_n(y)+\int_y^0\Pr\left(Y_1\leq y-z_n\mid\sigma\right)\mathrm{d}A_n(z_n)\end{aligned}$$

We know that:

$$\begin{aligned} A_n(z) &= \frac{1}{\sigma} \times a_n(z) \times \exp\left(\frac{z}{\sigma}\right) \\ &\Downarrow \\ \mathrm{d}A_n(z) &= \frac{1}{\sigma} \times \left[a_n'(z) + \frac{1}{\sigma} a_n(z)\right] \times \exp\left(\frac{z}{\sigma}\right) \end{aligned}$$

We can now plug this into the previous equation:

$$\begin{aligned} \Pr(Z_{n+1} \le y \mid \sigma) &= A_n(y) + \int_y^0 \Pr(Y_1 \le y - z_n \mid \sigma)\, \mathrm{d}A_n(z_n) \\ &= A_n(y) + \int_y^0 \exp\left(\frac{y - z_n}{\sigma}\right) \left[\frac{1}{\sigma}\left(a_n'(z_n) + \frac{1}{\sigma} a_n(z_n)\right) \exp\left(\frac{z_n}{\sigma}\right)\right] \mathrm{d}z_n \\ &= A_n(y) + \frac{1}{\sigma} \exp\left(\frac{y}{\sigma}\right) \int_y^0 \left(a_n'(z_n) + \frac{1}{\sigma} a_n(z_n)\right) \mathrm{d}z_n \\ &= \frac{1}{\sigma} a_n(y) \exp\left(\frac{y}{\sigma}\right) + \frac{1}{\sigma} \exp\left(\frac{y}{\sigma}\right) \int_y^0 \left(a_n'(z_n) + \frac{1}{\sigma} a_n(z_n)\right) \mathrm{d}z \\ &= \frac{1}{\sigma} \times \left[a_n(y) + \int_y^0 \left(a_n'(z) + \frac{1}{\sigma} a_n(z)\right) \mathrm{d}z_n\right] \times \exp\left(\frac{y}{\sigma}\right) \\ &= \frac{1}{\sigma} \times a_{n+1}(y) \times \exp\left(\frac{y}{\sigma}\right) \end{aligned}$$

Induction We can therefore conclude by induction that theorem 6.1 is true $\forall\, n$. $\square$

The analytical expression of a_n. We now solve analytically for a_n using the recursive formulation proved above.
We know:

$$\begin{aligned} a_{n+1}(y) &= a_n(y) + \int_y^0 \left(a_n'(z) + \frac{1}{\sigma} a_n(z)\right) \mathrm{d}z_n \\ &= a_n(y) + a_n(0) - a_n(y) + \frac{1}{\sigma} \int_y^0 a_n(z)\mathrm{d}z \\ &= a_n(0) + \frac{1}{\sigma} \int_y^0 a_n(z)\mathrm{d}z \end{aligned}$$

Since $A_n(0) = 1\ \forall n$, we have: $a_n(0) = \sigma\ \forall n$. So: $a_{n+1}(y) = \sigma + \frac{1}{\sigma} \int_y^0 a_n(z)\mathrm{d}z$.

We have:

$$\begin{aligned}
a_1(y) &= \sigma \\
a_2(y) &= \sigma + \frac{1}{\sigma}\int_y^0 [\sigma]\,\mathrm{d}z = \sigma - y \\
a_3(y) &= \sigma + \frac{1}{\sigma}\int_y^0 [\sigma - y]\,\mathrm{d}z = a_2(y) + \frac{1}{2\sigma}y^2 \\
a_4(y) &= \sigma + \frac{1}{\sigma}\int_y^0 \left[\sigma - y + \frac{1}{2\sigma}y^2\right]\mathrm{d}z = a_3(y) - \frac{1}{3!\sigma^2}y^3
\end{aligned}$$

Lemma 6.2.

$$a_n(y) = a_{n-1}(y) + (-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}y^{n-1}$$

Proof by recursion. Suppose:

$$a_n(y) = a_{n-1}(y) + (-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}y^{n-1}$$

Let us prove that this property holds at $n+1$. In other words, we want to show that:

$$a_{n+1}(y) = a_n(y) + (-1)^n\frac{1}{n!\sigma^{n-1}}y^n$$

We know that:

$$\begin{aligned}
a_{n+1}(y) &= \sigma + \frac{1}{\sigma}\int_y^0 a_n(z)\mathrm{d}z \\
&= \sigma + \frac{1}{\sigma}\int_y^0 \left[a_{n-1}(z) + (-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}z^{n-1}\right]\mathrm{d}z \\
&= \sigma + \frac{1}{\sigma}\int_y^0 [a_{n-1}(z)]\,\mathrm{d}z + \frac{1}{\sigma}\int_y^0 \left[(-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}z^{n-1}\right]\mathrm{d}z
\end{aligned}$$

So:

$$
\begin{aligned}
a_{n+1}(y) &= a_n(y) + \frac{1}{\sigma}\int_y^0 \left[(-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}z^{n-1}\right]\mathrm{d}z \\
&= a_n(y) + \frac{1}{\sigma}\left[(-1)^{n-1}\frac{1}{n}\frac{1}{(n-1)!\sigma^{n-2}}z^{n}\right]_y^0 \\
&= a_n(y) + (-1)^n\frac{1}{(n)!\sigma^{n-1}}y^n
\end{aligned}
$$

Induction We can therefore conclude by induction that the lemma is true $\forall\, n$. □

Analytical Expression. Using the recursive formula, we have:

$$
\begin{aligned}
a_{n+1}(y) &= a_n(y) + (-1)^n\frac{1}{n!\sigma^{n-1}}y^n \\
a_n(y) &= a_{n-1}(y) + (-1)^{n-1}\frac{1}{(n-1)!\sigma^{n-2}}y^{n-1} \\
\vdots &= \vdots \\
a_2(y) &= a_1(y) + (-1)^1\frac{1}{1!\sigma^{1-1}}y^1 \\
a_1(y) &= 0 + (-1)^0\frac{1}{0!\sigma^{0-1}}y^0
\end{aligned}
$$

Therefore:

$$
a_{n+1}(y) = \sum_{k=0}^{n}(-1)^k\frac{1}{k!\sigma^{k-1}}y^k
$$

Unconditional Posterior Distribution

Finally, using the prior ρ, we obtain:

$$
\Pr\left(\rho' \leq x\right) = \rho A_N\left(m(x,\rho);\ \sigma_L\right) + (1-\rho)A_N\left(m(x,\rho)\ ;\ \sigma_H\right)
$$

Conditional Posterior Distribution

We now characterize the conditional distribution of the posterior given a local shock. We know:

$$
\begin{aligned}
\rho'(\rho,\vec{\varphi}) &= \left[1 + \left(\frac{1-\rho}{\rho}\right)\times\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\times\left(\prod_{j=1}^{N}\left[1-\vec{\varphi}(j)\right]\right)^{\sigma_H^{-1}-\sigma_L^{-1}}\right]^{-1} \\
&= \left[1 + \left(\frac{1-\rho}{\rho}\right)\times\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\times\left(\prod_{k=1}^{N-1}\left[1-\vec{\varphi}(k)\right]\right)^{\sigma_H^{-1}-\sigma_L^{-1}}\left[1-\vec{\varphi}(j)\right]^{\sigma_H^{-1}-\sigma_L^{-1}}\right]^{-1}
\end{aligned}
$$

Let

$$\bar{x}(\rho,\varphi^j) \equiv \left[1+\left(1-\varphi^j\right)^{\sigma_H^{-1}-\sigma_L^{-1}}\left(\frac{1-\rho}{\rho}\right)\left(\frac{\sigma_H^{-1}}{\sigma_L^{-1}}\right)^N\right]^{-1}$$

Since $\prod_{k=1}^{N-1}\left[1-\vec{\varphi}(k)\right] \in [0,1]$, one can prove that $\rho'(\rho,\vec{\varphi}) \in (0,\ \bar{x}(\rho,\vec{\varphi}(j)))$. To characterize the distribution over $(0,\bar{x}(\rho,\varphi^j))$, let us consider $x \in (0,\bar{x}(\rho,\varphi^j))$.

$$\begin{aligned}
\Pr\left(\rho' \le x|\vec{\varphi}(j)=\varphi^j\right) &= \Pr\left\{\prod_{k=1}^{N}\left[1-\vec{\varphi}(k)\right] \le \left[\left(x^{-1}-1\right)\left(\frac{\rho}{1-\rho}\right)\left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)^N\right]^{\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}}}\right\} \\
&= \Pr\left\{\prod_{k=1}^{N-1}\left[1-\vec{\varphi}(k)\right] \le \frac{1}{1-\varphi^j}\left[\left(x^{-1}-1\right)\left(\frac{\rho}{1-\rho}\right)\left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)^N\right]^{\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}}}\right\}
\end{aligned}$$

Let:

$$\exp m(x,\varphi^j,\rho) \equiv \frac{1}{1-\varphi^j}\left[\left(x^{-1}-1\right)\times\left(\frac{\rho}{1-\rho}\right)\times\left(\frac{\sigma_L^{-1}}{\sigma_H^{-1}}\right)^N\right]^{\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}}} \in [0,1]$$

Hence, we have:

$$\begin{aligned}
\Pr\left(\rho' \le x|\vec{\varphi}(j)=\varphi^j\right) &= \Pr\left\{\prod_{k=1}^{N-1}\left[1-\vec{\varphi}(k)\right] \le \exp m(x,\varphi^j,\rho)\right\} \\
&= \rho\Pr\left[\sum_{k=1}^{N-1}\log\left[1-\vec{\varphi}(k)\right] \le m(x,\varphi^j,\rho)\mid\sigma_L\right] + \\
&\quad (1-\rho)\Pr\left[\sum_{k=1}^{N-1}\log\left[1-\vec{\varphi}(k)\right] \le m(x,\varphi^j,\rho)\mid\sigma_H\right] \\
&= \rho A_{N-1}\left(m(x,\varphi^j,\rho)\mid\sigma_L\right)+(1-\rho)A_{N-1}\left(m(x,\varphi^j,\rho)\mid\sigma_H\right)
\end{aligned}$$

The function $\mathscr{G}$ is the cumulative distribution function of the conditional posterior. This concludes the proof of Proposition 6. □

The probability density function is $g \equiv \mathscr{G}'$ and can be fully characterized.

Conditional Probability Density of Posterior Beliefs

By differentiating $\mathscr{G}$ with respect to x, we get:

$$g(x|\rho,\varphi^j) = \rho\left[A'_{N-1}\left(m(x,\varphi^j,\rho)|\,\sigma_L\right)m'(x)\right]+(1-\rho)\left[A'_{N-1}\left(m(x,\varphi^j,\rho)|\,\sigma_H\right)m'(x)\right]$$

To complete the analytical characterization, we also show that:

$$
\begin{aligned}
m(x,\varphi^j,\rho) &= -\log(1-\varphi^j) - \frac{1}{\sigma_L^{-1}-\sigma_H^{-1}} \log\left[\left(x^{-1}-1\right) \times \left(\frac{\rho}{1-\rho}\right) \times \left(\frac{\sigma_H}{\sigma_L}\right)^N\right] \\
m'(x,\varphi^j,\rho) &= -\frac{1}{\sigma_H^{-1}-\sigma_L^{-1}} \frac{1}{x(1-x)} \\
A_N(x;\sigma) &= \frac{1}{\sigma} a_N(x;\sigma) \exp\left(\frac{x}{\sigma}\right) \\
A_N'(x;\sigma) &= \frac{1}{\sigma}\left(a_N'(x;\sigma) + \frac{1}{\sigma} a_N(x;\sigma)\right) \exp\left(\frac{x}{\sigma}\right) \\
a_N(x;\sigma) &= \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{(k-1)!\sigma^{k-2}} x^{k-1} \\
a_N'(x;\sigma) &= \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{(k-2)!\sigma^{k-2}} x^{k-2} \ \text{ if } N \geq 2 \\
a_1'(x;\sigma) &= 0
\end{aligned}
$$